

THE EFFECTS OF TEACHERS' UNDERSTANDING OF ADDITION AND SUBTRACTION WORD PROBLEMS ON STUDENT UNDERSTANDING

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This study investigates the influence of teacher understanding on student understanding through teacher practice. Three elementary school teachers participated in a university course that discussed mathematical and pedagogical knowledge regarding addition and subtraction word problems. The data from this study were analyzed qualitatively to describe the nature of the teachers' understandings, the ways teachers used their understandings in their practice, and the nature of their students' understandings. I hypothesize that teachers used their understanding to create and implement tasks at a high level of cognitive demand, maintaining that demand over time, which affected student understanding.

There is quantitative evidence that content knowledge and pedagogical content knowledge influence student understanding (Hill, Rowan, & Ball, 2005; Fennema & Franke, 1992), but how teacher knowledge affects student achievement needs to be explored. In this study I begin to uncover how teacher knowledge affects student understanding using a qualitative perspective. If we assume that teacher understanding affects practice and that teacher practice affects student understanding (Hurry, Nunes, Bryant, & Pretzlik, 2005; Smith & Baker, 2001), then we can begin investigating these relationships.

Objectives

In three cases, I describe the nature of a teacher's understanding of a particular mathematical topic and investigate how that understanding affects the teacher's practice while teaching that topic. In addition, I examine the link between teacher understanding and student understanding in order to hypothesize about the relationships between them.

My study is exploratory in nature and seeks out possible connections that will need to be further developed. Since teacher knowledge is so expansive and varied, I limited this study to investigating one mathematical topic in light of only a few aspects of teacher knowledge. In this study I examine the understanding of three teachers regarding the content knowledge and pedagogical content knowledge shown to be integral to teaching addition and subtraction word problems, according to the Cognitively Guided Instruction (CGI) program (Carpenter, Fennema, Franke, Levi, & Empson, 1999). I investigate how a teacher's understandings of the structure of addition and subtraction word problems and of how students typically solve addition and subtraction word problems affect her practice, and how her practices affect student understanding of addition and subtraction.

Framework

My research uses Deborah Ball's construct of the mathematical knowledge needed for teaching (Ball, Phelps, & Thames, 2008). According to Ball, Specialized Content Knowledge (SCK) is an important aspect of subject matter knowledge and Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) are vital to pedagogical content knowledge. I have chosen to examine teacher and student understanding of addition and subtraction word problems due to the extensive research that has identified the content

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knowledge and pedagogical content knowledge for teachers in this area. I examine qualitatively how a teacher's understandings of the structure of addition and subtraction word problems, a substantial part of their SCK, affects students' understandings of addition and subtraction through its effects on teacher practice. I also examine how a teacher's understanding of the strategies students use to solve addition and subtraction word problems, part of their KCS as well as KCT, affects students' understandings of addition and subtraction through teacher practice.

I relied on research by Wiggins & McTighe and Sierpinska to establish when participants were exhibiting understanding. Understanding is "a family of interrelated abilities" that develops over time in an individual and involves "sophisticated insights and abilities, reflected in varied performances and contexts" (Wiggins & McTighe, 1998, p. 5). Wiggins and McTighe (1998) describe six indicators of understanding: *explanation, interpretation, application, perspective, empathy, and self-knowledge*. Sierpinska (1994) also identifies four indicators, which she calls mental operations, involved in understanding: *identification, discrimination, generalization, and synthesis*. There was some overlap in the two bodies of research, but each offered some valuable insight into indications of understanding. I referred to both Sierpinska's work and the work of Wiggins and McTighe to qualitatively describe the understandings of my participants.

Methodology

A pilot study indicated that the teachers in the local school district, unfortunately, did not have the type of content and pedagogical content knowledge I wanted to explore. Therefore, it was necessary to offer a course taught at a large university in the Mid-Atlantic region. The mathematics coordinator for the school district invited first-, second-, and third-grade teachers in the district to participate. Two first-grade and two second-grade teachers responded. The teachers in the study were selected from this course.

The course functioned as a discussion group with all four elementary school teachers, the mathematics coordinator, and myself. We met 10 times for a total of over 18 hours. The purpose of the course was to both improve and evaluate the teachers' understandings of how students think about addition and subtraction word problems, the classification of problems, the direct modeling of problems, the counting strategies used for solving problems, and the relationships among these topics, as researched by CGI. During the first half of the course I instructed the teachers. During the second half of the course the teachers decided how this knowledge could be used to change and enhance their own teaching. Throughout the last half of the course teachers were given the opportunity to evaluate their lesson plans that involved addition and subtraction word problems. The curriculum they used contained almost solely Join and Separate (Result Unknown) problems. They rewrote almost all the word problems in the curriculum, using each of the structure types identified by CGI.

Three teachers, Carmen, Pam, and Julie (all names used are pseudonyms), clearly exhibited understandings of the content and pedagogical content knowledge as described by current research, to observe in their classroom. I observed any class time during which the teachers taught addition and subtraction word problems. I audio-recorded the teaching sessions and took field notes, which included verbatim teacher and student interactions, writing on the board, or other relevant data not captured by the audio-recording. Copies of lesson plans and classwork were collected. The field notes, lesson plans, other additional documents such as student work, and transcribed audio-recordings were used to analyze the understandings teachers had, how this understanding affected the teachers' decision making, and how this understanding influenced student understanding. I also examined the students' understandings of the mathematics by listening to class discussions, teacher and student discussions, and by accessing student work.

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During interviews, the elementary school teachers in the study were asked about why they made specific decisions before and during class time. Events that merited attention during interviews included the teacher making changes to lesson plans, claims she made about student understanding during class, the teacher intervening in student work, assessments she made about students' solution strategies, or just conversations the teacher had with a student. I concentrated on observing four or five students in each classroom. Students were interviewed using mathematics tasks similar to those they encountered during their regular class work. To evaluate teacher and student understanding, I investigated participants' ability to *identify, explain, interpret, discriminate, apply, and generalize* within the context of understanding addition and subtraction word problems.

The analysis was conducted qualitatively using the data I collected in classroom observations, field notes, documents from the teachers, and interviews. The manner in which teachers used the curriculum provided me with an important avenue for investigating their understandings and how they utilized their understandings in practices such as planning lessons, choosing tasks, and anticipating student responses. I reviewed the data points searching for indicators of understanding in which I found multiple pieces of evidence. I also searched the data for evidence that teachers or students did not exhibit a specific understanding to make sure the assertions were made with as little bias as possible. Once I gained a better understanding of the understandings, I organized the categories into subcategories, based on types of understanding repeated in the data.

Next, I determined how the teacher used her understandings and how those understandings may have influenced the students' understandings. I usually worked backwards from a particular student understanding, investigating the data to find possible instances when the teacher may have influenced that understanding through her practice. In addition, I hypothesized how the teacher's knowledge could have affected the teacher's practice. Multiple instances of a particular student understanding, coupled with multiple instances of a particular practice influencing that understanding, merited a connection worth noting.

Results

The following paragraphs suggest some aspects of student understanding influenced by teacher understanding as well as aspects of teacher practice that influenced student understanding.

Aspects of Student Understanding Influenced by Teacher Understanding

Student understanding of their written representations was one of the most notable items influenced by the teachers' understandings. Students understood that addition and subtraction word problems could be represented with devices such as equations, pictures, diagrams, or tally marks. Teachers influenced how the students were able to interpret the representations and how they used the representations to explain and interpret the problems given to them. For example, Pam asked students to write an equation to represent each problem they were asked to solve. She used equation writing as a tool for students to observe and articulate the differences in structures. Her consistent emphasis on where the question mark was in the equation enabled students to show what parts of the problem were given and what part of the problem was missing. Similarly, the teachers influenced the quality of students' verbal explanations of their thinking. All three teachers used written and verbal representations that described, as literally as possible, the thoughts and actions of a typical student when solving the problems (as described by CGI), which may have contributed to why the practices were relatively easily incorporated by their

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students.

There was an extensive amount of evidence in all three classrooms that teacher understanding can lead to an expanded student understanding of mathematical operations. I believe that the teachers' flexible understandings enabled the students to interpret the mathematical operation as a strategy used to solve the problem instead of a defining characteristic of the problem. The students could focus on the problems' structure, not relying on an operation to interpret problem. The operations were possible avenues for solving the problems. The teachers asked the students to solve problems in multiple ways, including by the use of different operations. Students showed an ability to interpret the operations as "opposites" of each other. They explained that the opposite operation could be used to check their work or to solve a problem using a second method.

Students in this study were able to learn about concepts such as recognizing invariance, generalizing, justifying, and focusing on the structure of the problem instead of just how to solve a problem. For example, when one student, Niki, solved a Join (Change Unknown) she wrote the equation $32 + ? = 50$. Niki then wrote $30 + 20 = 50$ and then $32 + 18 = 50$. Her teacher asked her about her thinking. The student responded:

Niki: Because this [the number in the problem] is 32, this [the number in her equation] has to be 32. So I added 2 more. Then I took 2 less from the 20 because if I added 2 more to here [the first addend] I have to take away from there [the second addend].

Pam: How did you know the 50 would not change?

Niki: Because I added 2 more and took away 2 more so it would be the same thing.

Niki showed surprising mathematical insight with her generalization and her nascent understanding that she could add to one addend and subtract from the other addend and maintain the same sum. These young students were developing the building blocks for identifying and explaining these important mathematical practices and powerful mathematical concepts.

The ability of students to progress to more sophisticated levels of problem solving was also influenced by the teachers' mathematical and pedagogical understandings. The teachers understood the students' current ability levels and they also understood the more sophisticated strategies, so they developed tasks to build on the students' understanding. The teachers understood the importance of students developing their individual interpretations of the mathematical topic. Practices such as refraining from teaching a standard algorithm until the students were capable of understanding the algorithm allowed students to develop their own strategies as shortcuts for direct modeling and counting strategies. All three teachers encouraged their students to use less sophisticated methods for solving problems when they believed a student lacked the ability to interpret the nature of the problem, which the teachers hoped would lead to a deeper mathematical understanding for the student.

Aspects of Teacher Practice That Influenced Student Understanding

The first and most drastic change in the teacher's practice was the complete revamping of the problems given to the students. It is clear that the teachers' understandings affected their practice through the types of problems they chose to give to their students and through the sequencing of those problems. The teachers also used their understandings to comprehend and carry out the objectives for each of their lessons. They were able to make more specific lesson objectives and have those objectives guide their interactions with the students throughout their teaching. They could decide whether the purpose of their lessons was to introduce a certain type of problem, develop interpretations for problems, discriminate among different types of problems, or develop strategies to solve problems.

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The teachers' understandings enabled them to better address the needs of their students, not only through the building of their lesson plans but also in daily interactions with students. Prior to this study, the teachers were unhappy with how they taught the lessons, but were unable to make changes because they did not understand the mathematics well enough. By their own admissions, their knowledge only enabled them to change the numbers in the problem to increase or decrease difficulty. Because of their increased mathematical understandings they were able to construct tasks that would build on student understandings. The teachers gained insight into students' weaknesses and strengths, and could modify their practice accordingly. For example, Julie noticed that a group of two higher achieving students answered all of their problems correctly except for the Compare (Referent Unknown) problems. She was able to design more problems of this type and target her verbal questions to address the relationships involved.

All three teachers identified that the questions they asked their students were more targeted than in previous years on the mathematics they wanted their students to understand, because they understood the mathematics better. For example, the teachers' understandings of structure were evident in their teaching when they continuously asked students to explain the relationships within the word problems. Carmen made a practice of asking students to explain the story without using any numbers. These teachers did not simply expect students initially to be able to explain the nature of the problem; they used scaffolding to press students to interpret the numbers in the problem, the objects to which those numbers refer, and the relationships among the referents within the problem, instead of focusing on getting an answer or which operation to use to solve the problem.

The teachers used multiple representations in their teaching, such as blocks, pictures, equations, and diagrams. Teachers asked students to model high-level performance with explanations for how they wrote their equations, including interpreting why they decided to represent the quantities in the manner they chose and where to place the quantities in the equation. The teachers encouraged the use of direct modeling to understand the nature of difficult problems. The teachers spent time not only showing examples of ways to solve problems but also explaining how the strategies worked and how the strategies were contingent upon the nature of the problems they were attempting to solve. Asking the students to solve problems in multiple ways increased the flexibility of their understanding (Star & Rittle-Johnson, 2008). It gave the students further incentive and avenues to develop their strategies and develop their ability to explain and interpret the mathematics they used.

Finally, teachers were able to find validity in the arguments students made that were different from their own, and encouraged students to continue in their thinking. For example, Pam found herself wanting to ask students to solve Separate (Change Unknown) problems using the missing addend approach, because that is how she conceptualizes those problems. But she allowed her students to develop their own strategies, based on their interpretation. She identified that there are many appropriate ways to think about mathematical problems, and decided not to instruct her students specifically how to solve them. Teachers can allow students to develop their correct interpretations, merely by knowing when to refrain from imposing their own ideas that might hinder productive student thinking.

Discussion

After analyzing the data, I hypothesize that teachers' mathematical knowledge for teaching determines the teacher's ability to design and implement a task at a high level of cognitive demand as well as their ability to maintain that level throughout implementation. The following figure adapted from Henningsen and Stein (1997) illustrates the three phases through which tasks

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pass; the task in the curriculum, the task set up by the teacher, and the task as implemented in the classroom. The way the task is implemented in the classroom directly affects students' learning outcomes.

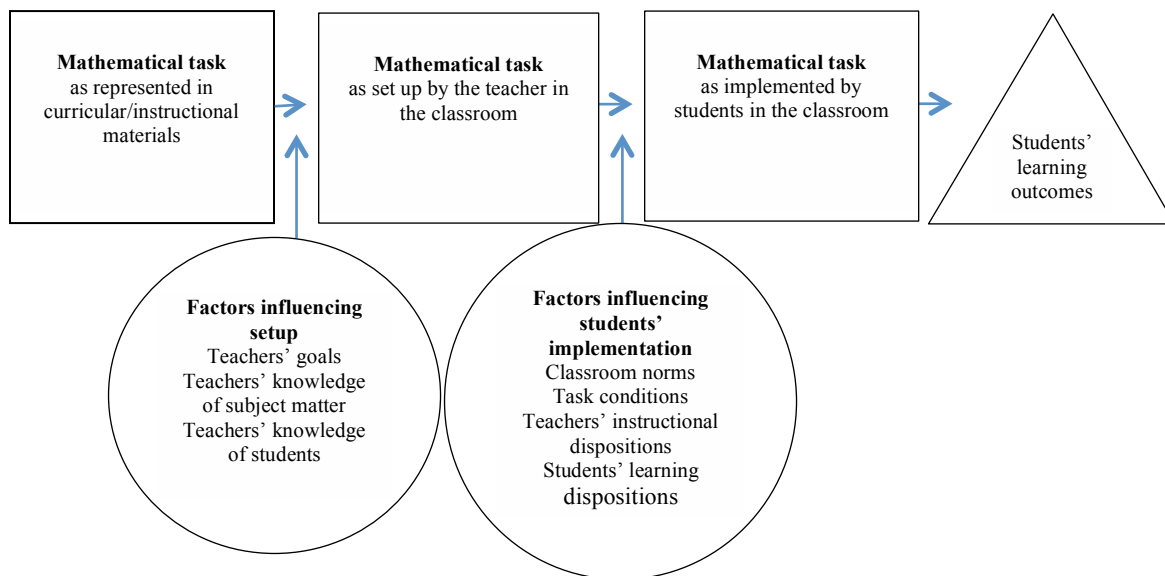


Figure 2. Framework for Mathematical Tasks (adapted from Henningsen & Stein, 1997, p. 528)

Research shows that the level of cognitive demand of a task tends to decrease as the task proceeds through the framework. In the framework, teacher knowledge of subject matter appears as a factor influencing the setup of the task, but it does not appear as a factor influencing the implementation of the task. I agree that teachers' instructional dispositions determine the way a teacher desires to implement a task, but, a cross-case analysis of the participants in my study showed that not only did they desire to implement tasks at a high level of cognitive demand – but they also used their understanding to create situations in which students were using tasks at a high level of cognitive demand and maintaining that demand throughout implementation.

The teachers used their understanding to create the two types of higher level cognitive demand tasks, “using procedures with connections” and “doing mathematics” indicated by Henningsen and Stein. Tasks that use procedures with connections were chosen to “focus students’ attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas” (Stein, Smith, Henningsen, & Silver, 2000, p. 16). For example, Carmen asked her students to use the procedure of answering a list of questions she constructed to encourage them to think about the relationships between the quantities in the word problems, such as “Who has more?” “What clue tells you this?” etc. The students followed this procedure trying to make connections among the numbers in the problem, the quantities they represented, and the relationship between those quantities. The teachers also used problems that asked students to represent the quantities and actions in the problem in multiple ways, such as with diagrams, manipulatives, symbols, and problem situations (Stein et al., 2000, p. 16). Because the teachers were aware of the connections they were able to incorporate those connections with procedures. Teachers consistently asked students to make connections among the representations to develop their meaning, such as when Pam asked students to write mathematical equations to lead students to understand the types of knowns and

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unknowns in the addition and subtraction word problems.

The teachers also used their understanding to create higher-level cognitive demand tasks called “doing mathematics.” Instead of giving students the same type of problem repeatedly and telling them how to solve it, students were given problems that required nonalgorithmic thinking. Students were purposefully not given algorithms so they could develop more complex thinking about the operations. Teachers thought flexibly about the problems—not defining them by a single operation,—allowing different operations, depending on the strategies students chose. All three teachers chose tasks that required considerable cognitive effort so that some level of anxiety was experienced by the students at some time during their work, which was a change from previous years when the students were described as “bored.”

The teachers were also able to use their understanding to maintain that demand level over time. According to Stein and colleagues there are seven factors associated with maintenance of high-level cognitive demands (Stein et al., 2000). These factors coincide with the aspects of teacher practice that I claimed to be influenced by teacher understanding. I will discuss four of the factors I saw evidenced by the teachers in this study.

Eighty-two percent of tasks that stayed at a high cognitive demand were tasks that built on students’ prior knowledge (Henningsen & Stein, 1997). The participants in my study decided to write a pretest for their students. Through this test the teachers realized the students were capable of much more than the teachers were previously expecting. The teachers knew the demand of the tasks in the current curriculum was too low, but they did not have the understanding necessary to change them. During this study they completely changed their curriculum, based on the results of the pretest they gave their students. The teachers also encouraged the students to use strategies to solve problems based on their understanding of the students’ ability levels. Using their increased mathematical understandings, teachers were better able to determine the students’ prior understandings and design activities that could improve those understandings—without their increased mathematical understanding they could only increase or decrease the size of the numbers.

Scaffolding of student thinking and reasoning is another significant factor to maintaining tasks’ high level of cognitive demand. With each teacher I witnessed situations in which students were unable to complete a task and the teacher provided assistance through questioning and comments that asked students to think about the structure of the problem without reducing the demand of the task. For example, when one student was unable to understand a Separate (Change Unknown) problem, Carmen helped her develop a strategy using tally marks to solve the problem. The student was eventually able to build off this interaction and solve these problems without Carmen’s assistance.

Carmen also demonstrated a fourth factor, giving students a means of monitoring their own progress. She gave the students a checklist to fill out for each problem to enable students to guide their own progress in problem solving. 1) Read the problem. 2) Look for clues in the problem, find the important information. 3) Think of some strategies. 4) Solve the problem. 5) See if it makes sense. Carmen reported that this type of list was unnecessary before because the students “didn’t have to think” about the problems. Now that her curriculum did contain problem solving that required students’ cognitive effort, she addressed the development of students’ ability to monitor each step toward solving the problem.

These teachers modeled high-level solution strategies, another tool for effective maintenance of high cognitive demand. They explained how solution strategies were appropriate based on the nature of the problems. They acknowledged that they were unable to do this successfully before

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learning more about addition and subtraction word problems. They were aware of the important relationships in problems and could therefore model these relationships in a variety of ways. Part of these demonstrations involved drawing conceptual connections, another important factor in maintaining the task demand level. For example, teachers could draw connections between addition and subtraction when demonstrating the use of both of the operations within one word problem.

I suggest that student understanding was affected by the teachers' maintenance of the level of cognitive demand. The students demonstrated the type of understandings fostered by tasks with higher cognitive demand level that focused on "procedures with connections" (Stein et al., 2000). Students were able to develop deeper levels of understanding of mathematical concepts, such as their expanded understanding of the operations of addition and subtraction, through their use of procedures. They represented problems in multiple ways, such as modeling, equation writing, and creating their own word problems. Students were able to develop meaning by making connections among multiple representations. For example, the students in Pam's class were able to use equation writing, picture drawing, and creating of problem situations to better understand the relationships between types of knowns and unknowns in an addition and subtraction word problem. There were many examples of students showing cognitive effort, not following procedures mindlessly, and engaging with underlying conceptual ideas. There is also evidence that students were able to demonstrate the type of understandings that "doing mathematics" tasks are intended to engender. Students showed nonalgorithmic thinking while solving the addition and subtraction problems in multiple ways. There is evidence of students beginning to explore aspects of mathematical concepts such as invariance, generalizing, justifying, and focusing on the structure of the problem instead of just how to solve a problem. There were countless examples of students accessing relevant knowledge and using it appropriately to work through a task.

The teachers each expressed that they knew they were not effectively teaching this topic prior to the study. They did not realize they were lacking the mathematical understanding necessary to enhance their practice. Within days of increasing their SCK, KCS, and KCT they were able to change their practice. Their teaching practices, affected by their deeper understanding, influenced the quality of their students' understandings.

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